A Modified MoL Algorithm with Faster Convergence and Improved Computational Efficiency

by

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Abstract

This paper presents an improved algorithm for the method of lines (MoL) which converges much faster than the conventional method. While the error in the MoL is typically o(h^2), the error in the modified MoL is reduced to o(h^4). Therefore, accuracy can be maintained with a much smaller number of lines leading to reduced matrix sizes and thus accelerating the algorithm considerably. Numerical results obtained for a microstrip line illustrate the advantages of this new idea.

Introduction

The MoL is well established as a versatile numerical tool for the analysis of electromagnetic field problems in microwave, millimeter wave and integrated optic circuits (i.e. [1]-[9]). Due to its semi-analytical approach the computational effort is much less than for other methods applied to the same problem. Furthermore, since this method requires no basis functions, relative convergence phenomena can be avoided.

A disadvantages of this method, in particular when multiconductor transmission lines are analyzed or when very fine circuit structures must be resolved, is that it is difficult to find the appropriate number of lines to satisfy the boundary conditions and edge condition simultaneously. One way to overcome this problem is to increase the number of lines, which would also increase the matrix sizes and therefore makes this algorithm computationally very inefficient and potentially unstable [11]. Another way to treat this problem is to introduce nonequidistant discretization [5]. This approach allows smaller matrix sizes and improves the computational efficiency of the method. In both cases, however, the error remains in the order of o(h^2), with h being the spacing between the lines.

In this paper a modified approach to the method of lines is introduced which reduces the error to o(h^4). This means that the convergence rate is much faster compared to the conventional MoL. Or, in other words, to achieve the same accuracy known from the conventional approach [1], a smaller number of lines is required.

To appreciate the modifications made, one must understand that the overall error in the MoL results from the discretizations of the Helmholtz equation, the continuity equation and the edge condition. Any modification in only one of these sources of error will not lead to a reduction of the overall error. Therefore, we introduce modifications in each source of error.

Discretization error in the Helmholtz equation

The discretization error in the Helmholtz equation has been investigated by [12], although the original idea for this approach was given already in 1939 in [10], and similarly in [13] and [11]. The discretized Helmholtz equation using the three neighbouring lines leads to the following expression :

$$\frac{1}{12}\frac{\partial^2}{\partial y^2}(\psi_{i+1}^{e,h}+10\psi_i^{e,h}+\psi_{i-1}^{e,h})+(\psi_{i+1}^{e,h}-2\psi_i^{e,h}+\psi_{i-1}^{e,h})/h^2$$

$$+\frac{1}{12}(k^2-\beta^2)(\psi_{i+1}^{e,h}+10\psi_i^{e,h}+\psi_{i-1}^{e,h})=r_{i0}^{e,h},\quad i=1,\ 2,\ \ldots,\ N \tag{1}$$

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Developing (1) into a Taylor's series shows clearly that the error is now reduced to o(h^4)

$$r_{i0}^{e,h} = \frac{7}{720} h^4 \frac{\partial^6}{\partial x^6} \psi_i^{e,h} + o(h^6) \qquad (2)$$

The discretized Helmholtz equation in matrix form is therefore:

$$h^2[Q]\frac{d^2}{dy^2}\overline{\psi}^{e,h} - [P]\overline{\psi}^{e,h} + h^2(k^2-\beta^2)[Q]\overline{\psi}^{e,h} = 0 \qquad (3)$$

For the following derivations, the remaining error may be neglected. The difference to the conventional MoL is that instead of only a tridiagonal matrix [P] there is now an additional matrix [Q] to be considered in (3), which is different from [P]. In other words, to find an analytical solution for this equation, one must not only diagonalize matrix [P] but also matrix [Q] with the same transformation matrix. For the N-D boundary condition the following transformation matrix has been found, which is different from the one utilized in the conventional MoL

$$[T]_{ij} = 2\sqrt{\frac{5+\cos\varphi_j}{N+1/2}}\cos(i-1/2)\varphi_j,\ \varphi_j = \frac{j-1/2}{N+1/2} \qquad (4)$$

Multiplying (3) from the left and the right with (4) leads to a decoupled ordinary differential equation which can be solved analytically

$$h^2\frac{d^2}{dy^2}V_i^{e,h} - [\lambda_i - h^2(k^2-\beta^2)]V_i^{e,h} = 0 \qquad (5)$$

with $[\lambda]$ being a diagonal matrix

$$\lambda_i = \mu_i/(3-\mu_i),\ \mu_i = \sin^2\varphi_j/2 \qquad (6)$$

and $V_i^{e,h}$ being the transformed potential functions. This is a more general procedure which allows to diagonalize the normal matrices [P] and [Q] if at least one of them is positive definite.

Discretization error in the continuity equation

To minimize the error also in the continuity equation at the interface between two different transmission media k and (k+1), the fields must be modified in the following way (E_x field as example):

$$\frac{\beta}{\omega\varepsilon_0}\{(\frac{\psi^e_{k,i+1}-\psi^e_{k,i}}{h} - r^e_{0i,k})/\varepsilon_{rk} - (\frac{\psi^e_{k+1,i+1}-\psi^e_{k+1,i}}{h} - r^e_{0i,k+1})/\varepsilon_{rk+1}\}$$

$$= \frac{1}{24}\frac{d}{dy}\{(\psi^h_{k+1,i+1}-\psi^h_{k,i+1}) + 22(\psi^h_{k+1,i}-\psi^h_{k,i}) + (\psi^h_{k+1,i-1}-\psi^h_{k,i-1})\} \qquad (7)$$

similarly for the other field components. Again, analyzing this equation leads to a remaining error of o(h^4)

$$r^e_{0i,k,k+1} = \frac{17}{2\cdot 4!\cdot 5!} h^4 \frac{\partial^5}{\partial x^5}\psi^e_{k,k+1,i} + o(h^6) \qquad (8)$$

And thus (7) can be written as

$$\frac{\beta}{\omega\varepsilon_0}[D_x]\{\overline{\psi}^e_k/\varepsilon_{rk} - \overline{\psi}^e_{k+1}/\varepsilon_{rk+1}\} = [Q_x]\frac{\partial}{\partial y}\{\overline{\psi}^h_{k+1} - \overline{\psi}^h_k\} \qquad (9)$$

Edge parameter

In the above error analysis it is assumed that the higher order partial derivatives of the potential functions are continuous. However, the field behaviour is singular once it approaches the edges of the conductor. The fields must satisfy the edge condition which, in the conventional MoL, minimizes the discretization error if the last line on the conductor is 0.265 h away from the edge [6]. This is usually approximated by 0.25 h. In the modified MoL approach the optimum distance is found to be 0.30h.

Numerical Results

To illustrate the accuracy and improved convergence behaviour of the modified MoL, we have analyzed shielded microstrip lines with different geometries. Fig. 1 shows the effective dielectric constant of the quasi-TEM mode versus frequency. The dispersion diagram shows that with only 2 lines on the strip (assuming symmetry) and 4 lines on the dielectric interface we achieve the same results as Pregla [1] with 3 lines on the strip and 6 in the dielectric interface. The same tendency can be observed from Fig.2 for the coupled microstrip lines. The modified MoL requires only 2 lines on the conductor and 9 lines in the dielectric interface while the conventional MoL [1] requires 3 lines on the strip and 13 in the dielectric interface. For structures which usually require a higher resolution in the conductor area, because the relative strip dimensions are small compared to the overall structure, the modified MoL is clearly an improvement over the conventional approach. This is shown in Fig.3 where convergence is almost reached by using only 2 lines on the conductor while the conventional MoL requires more than 6 lines to reach the same results. In a structure with wider strip dimensions and smaller enclosure, the difference between the conventional and the modified MoL is not so significant, as shown in Fig.4. Therefore, it is expected that the modified MoL will show its full potential in conjunction with the nonequidistant discretization scheme in order to resolve extremely fine circuit details. In those cases it becomes extremely important to reduce matrix sizes in every step of the computation since the complexity of the structure requires a considerable number of lines.

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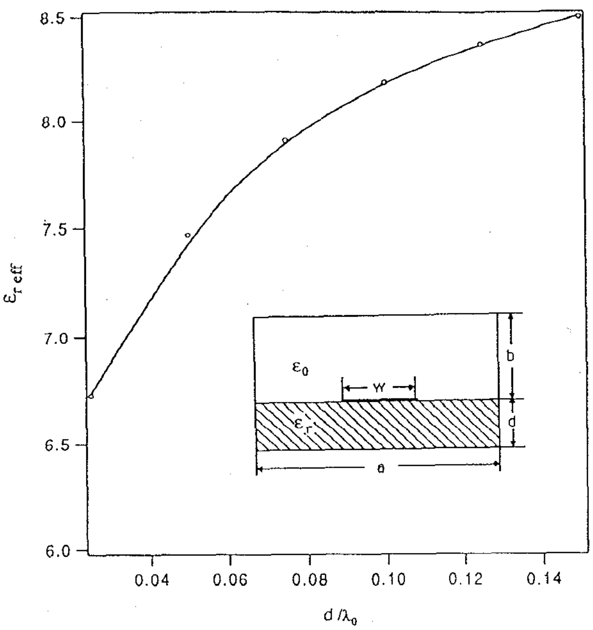

Fig. 1 Effective dielectric constant of a single microstrip line
$w/d = 2.0$, $a/d = 7.0$, $b/d = 3.0$, $\varepsilon_r = 9.0$

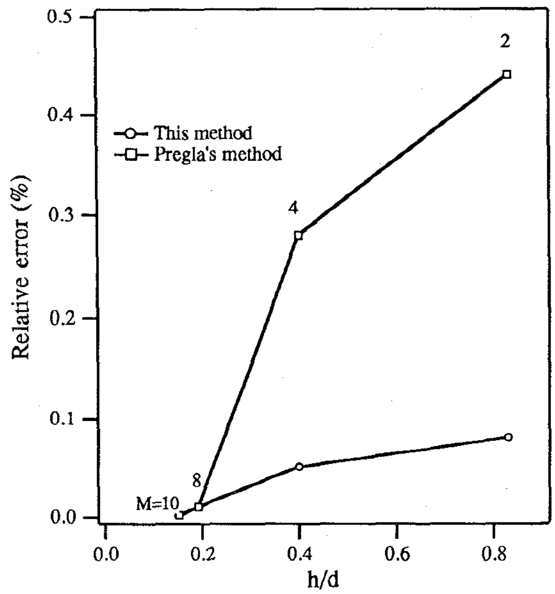

Fig. 3 Relative errors for the microstrip line
$a/w = 7.5$, $w/d = 3.0$, $b/d = 100$, $d/\lambda_0 = 0.02$, $\varepsilon_r = 10$

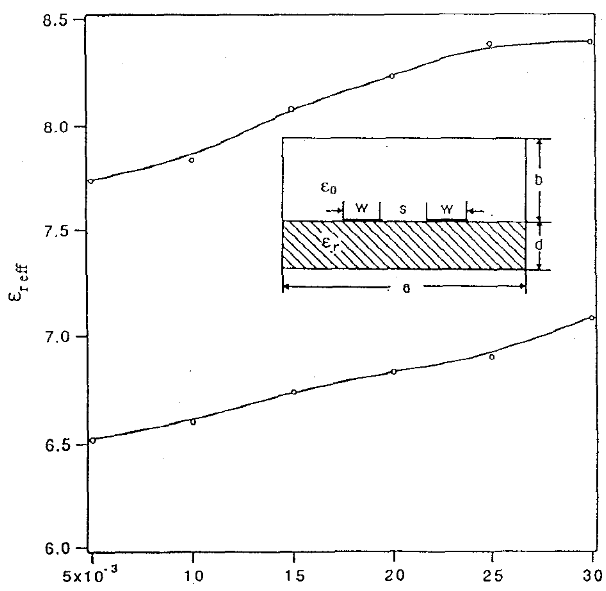

Fig. 2 Effective dielectric constant of coupled microstrip lines
$w/d = 1.5$, $s/d = 1.5$, $a/d = 20$, $b/d = 19$, $\varepsilon_r = 10.2$

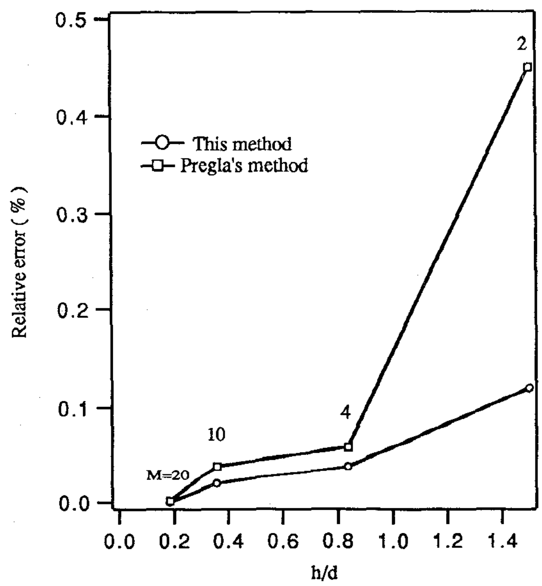

Fig. 4 Relative errors for the microstrip line
$a/w = 2.5$, $w/d = 3.0$, $b/d = 30$, $d/\lambda_0 = 0.02$, $\varepsilon_r = 10$